

On semiabelian categories in Functional Analysis and Topological Algebra

Dinamérico P. Pombo Jr.

Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530
21945-970 Rio de Janeiro, RJ Brasil
dpombo@terra.com.br

Abstract. In this work we discuss an elementary self-contained presentation of the notion of a semiabelian category, introduced by Raikov and Palamodov. Fundamental examples of (non-abelian) semiabelian categories occurring in Functional Analysis and Topological Algebra are treated in detail throughout the paper.

§1. Introduction

Although abelian categories offer an adequate framework for dealing with relevant aspects of Algebraic Geometry and Homological Algebra, they do not contain many important categories which arise in Functional Analysis and Topological Algebra, such as the categories of Banach spaces, locally convex spaces, abelian Hausdorff topological groups, topological modules over a topological ring and linearly topologized modules over a topological ring. In [26], Raikov introduced the so-called semiabelian categories (which were extensively studied in [31]), which contain all abelian categories as well as certain categories occurring in Functional Analysis and Topological Algebra which are not abelian. Subsequently, Palamodov considered the notion of a semiabelian category in a more gen-

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eral form (see [3,28,29,36] for justifications of this assertion) and studied its application to certain topics of the theory of locally convex spaces.

In this paper we discuss the concept of a semiabelian category in the sense proposed by Palamodov. In order to make the exposition accessible to non-specialists in Homological Algebra, we recall in the second section a few basic notions and facts concerning general categories, the notion of a category being a creation of Eilenberg and Mac Lane [11]. At this point we have avoided the customary use of the axiom of choice in defining subobjects and quotients, which are important for our purposes, following an approach due to Gabriel. The third section is devoted to basic facts about preadditive categories which are needed for the comprehension of the last section. Finally, in the last section, we use the background presented in the preceding ones to introduce semiabelian categories. After establishing some preparatory results, we concentrate our efforts in proving a general criterion of bijectivity and an isomorphism theorem in the context of semiabelian categories, in whose statements the concept of a strict morphism plays a central role and which readily imply the known isomorphism theorems for abelian categories. It should be mentioned that various examples of relevant semiabelian categories which are not abelian, encountered in Functional Analysis and Topological Algebra, are worked out in detail throughout the text.

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§2. Preliminaries on categories and examples

For each *category* \mathcal{C} [12,14], $\text{Ob}(\mathcal{C})$ will denote the class of *objects* in \mathcal{C} and, for $A, B \in \text{Ob}(\mathcal{C})$, $\text{Mor}_{\mathcal{C}}(A, B)$ will denote the set of *morphisms* from A into B in \mathcal{C} . An element u of $\text{Mor}_{\mathcal{C}}(A, B)$ will be represented by $u: A \rightarrow B$ or $A \xrightarrow{u} B$. For $A \in \text{Ob}(\mathcal{C})$, the identity morphism of A will be represented by 1_A .

Let us mention some examples of categories.

Example 2.1. The category Set whose objects are the sets where, for $A, B \in \text{Ob}(\text{Set})$, $\text{Mor}_{\text{Set}}(A, B)$ is the set of mappings from A into B .

Example 2.2. The category Top whose objects are the topological spaces where, for

$A, B \in \text{Ob}(\text{Top})$, $\text{Mor}_{\text{Top}}(A, B)$ is the set of continuous mappings from A into B .

Example 2.3. The category Grp whose objects are the groups where, for $A, B \in \text{Ob}(\text{Grp})$, $\text{Mor}_{\text{Grp}}(A, B)$ is the group of group homomorphisms from A into B .

Example 2.4. For each ring R with a non-zero identity element, we can consider the category Mod_R whose objects are the unitary left R -modules [5,20] where, for $A, B \in \text{Ob}(\text{Mod}_R)$, $\text{Mor}_{\text{Mod}_R}(A, B)$ is the additive group of R -linear mappings from A into B . In the special case where R is a field K (resp. R is the ring \mathbb{Z} of integers), Mod_R is the category of vector spaces over K (resp. the category of abelian groups).

Example 2.5. For each topological ring R with a non-zero identity element, we can consider the category Topm_R whose objects are the unitary left topological R -modules [34] where, for $A, B \in \text{Ob}(\text{Topm}_R)$, $\text{Mor}_{\text{Topm}_R}(A, B)$ is the additive group of continuous R -linear mappings from A into B . In the special case where R is a topological field K (resp. R is the ring \mathbb{Z} of integers endowed with the discrete topology), Topm_R is the category of topological vector spaces over K [34] (resp. the category of abelian topological groups [4,34]).

Example 2.6. For each topological ring R with a non-zero identity element, we can consider the category Ltm_R whose objects are the unitary left linearly topologized R -modules [23,34] where, for $A, B \in \text{Ob}(\text{Ltm}_R)$, $\text{Mor}_{\text{Ltm}_R}(A, B)$ is the additive group of continuous R -linear mappings from A into B . In the special case where R is a discrete field K , Ltm_R is the category of linearly topologized spaces over \mathbb{K} (in the current literature [17,21], linearly topologized spaces are also assumed to be Hausdorff spaces).

Example 2.7. The category Ahtg whose objects are the abelian Hausdorff topological groups where, for $A, B \in \text{Ob}(\text{Ahtg})$, $\text{Mor}_{\text{Ahtg}}(A, B)$ is the abelian group of continuous group homomorphisms from A into B .

Example 2.8. The category Ban whose objects are the (real or complex) Banach spaces [17,27] where, for $A, B \in \text{Ob}(\text{Ban})$, $\text{Mor}_{\text{Ban}}(A, B)$ is the (real or complex) vector space of continuous linear mappings from A into B .

Example 2.9. The category \mathbf{Lcs} whose objects are the (real or complex) locally convex spaces [17,27] where, for $A, B \in \mathbf{Ob}(\mathbf{Lcs})$, $\mathbf{Mor}_{\mathbf{Lcs}}(A, B)$ is the (real or complex) vector space of continuous linear mappings from A into B .

Example 2.10. For each complete non-Archimedean non-trivially valued field K , we can consider the category \mathbf{Lcs}_K whose objects are the locally K -convex spaces [30,33] where, for $A, B \in \mathbf{Ob}(\mathbf{Lcs}_K)$, $\mathbf{Mor}_{\mathbf{Lcs}_K}(A, B)$ is the vector space over K of continuous K -linear mappings from A into B .

Example 2.11. For each topological ring R with a non-zero identity element, we can consider the category \mathbf{Borm}_R whose objects are the bornological R -modules [24] where, for $A, B \in \mathbf{Ob}(\mathbf{Borm}_R)$, $\mathbf{Mor}_{\mathbf{Borm}_R}(A, B)$ in the additive group of bounded R -linear mappings from A into B . In the special case where R is a topological field K , $\mathbf{Mor}_{\mathbf{Borm}_R}(A, B)$ is the category of bornological vector spaces over K (the case where K is a complete non-trivially valued field was considered in [32]).

Example 2.12. The category \mathbf{Cbvs} whose objects are the (real or complex) convex bornological vector spaces [16] where, for $A, B \in \mathbf{Ob}(\mathbf{Cbvs})$, $\mathbf{Mor}_{\mathbf{Cbvs}}(A, B)$ is the (real or complex) vector space of bounded linear mappings from A into B .

Example 2.13. For each complete non-Archimedean non-trivially valued field K , we can consider the category \mathbf{Cbvs}_K whose objects are the K -convex bornological vector spaces [1] where, for $A, B \in \mathbf{Ob}(\mathbf{Cbvs}_K)$, $\mathbf{Mor}_{\mathbf{Cbvs}_K}(A, B)$ is the vector space over K of bounded K -linear mappings from A into B .

Definition 2.14. Let \mathcal{C} be a category. The *dual category* of \mathcal{C} , denoted by \mathcal{C}^o , is defined as follows: (a) $\mathbf{Ob}(\mathcal{C}^o) = \mathbf{Ob}(\mathcal{C})$; (b) for $A, B \in \mathbf{Ob}(\mathcal{C}^o)$, $\mathbf{Mor}_{\mathcal{C}^o}(A, B) = \mathbf{Mor}_{\mathcal{C}}(B, A)$. The composition of morphisms

$$\mathbf{Mor}_{\mathcal{C}^o}(A, B) \times \mathbf{Mor}_{\mathcal{C}^o}(B, C) \longrightarrow \mathbf{Mor}_{\mathcal{C}^o}(A, C)$$

in \mathcal{C}^o is defined by the composition of morphisms

$$\mathbf{Mor}_{\mathcal{C}}(C, B) \times \mathbf{Mor}_{\mathcal{C}}(B, A) \longrightarrow \mathbf{Mor}_{\mathcal{C}}(C, A)$$

given in \mathcal{C} . Clearly, $(\mathcal{C}^o)^o = \mathcal{C}$.

Definition 2.15. Let \mathcal{C} be a category and $u \in \text{Mor}_{\mathcal{C}}(A, B)$. u is said to be *injective* (resp. *surjective*) if, for all $C \in \text{Ob}(\mathcal{C})$, the mapping $v \in \text{Mor}_{\mathcal{C}}(C, A) \mapsto uv \in \text{Mor}_{\mathcal{C}}(C, B)$ (resp. $w \in \text{Mor}_{\mathcal{C}}(B, C) \mapsto wu \in \text{Mor}_{\mathcal{C}}(A, C)$) is injective; u is said to be *bijective* if it is injective and surjective; u is said to be an *isomorphism* if there exists a $v \in \text{Mor}_{\mathcal{C}}(B, A)$ such that $uv = 1_B$ and $vu = 1_A$ (v is necessarily unique and denoted by u^{-1}).

Two objects D, E in \mathcal{C} are said to be *isomorphic* if there is an isomorphism $w: D \rightarrow E$ in \mathcal{C} .

It is easily seen that the following assertions hold:

- (a) In order that $u \in \text{Mor}_{\mathcal{C}}(A, B)$ be injective (resp. $u \in \text{Mor}_{\mathcal{C}}(A, B)$ be surjective), it is necessary and sufficient that $u \in \text{Mor}_{\mathcal{C}^o}(B, A)$ be surjective (resp. $u \in \text{Mor}_{\mathcal{C}^o}(B, A)$ be injective).
- (b) If $u \in \text{Mor}_{\mathcal{C}}(A, B)$, $v \in \text{Mor}_{\mathcal{C}}(B, C)$ and vu is injective (resp. vu is surjective), then u is injective (resp. v is surjective).

As a consequence of (b), every isomorphism is bijective. The following example furnishes a bijective morphism which is not an isomorphism.

Example 2.16. Let A be the additive group of real numbers endowed with the discrete topology and B the additive group of real numbers endowed with the usual topology, and let $u: A \rightarrow B$ be given by $u(x) = x$ for $x \in A$. Then $u \in \text{Mor}_{\text{Ahtg}}(A, B)$ is bijective, but u is not an isomorphism.

Definition 2.17 [13]. Let \mathcal{C} be a category and $A \in \text{Ob}(\mathcal{C})$. A *subobject* of A is a pair (A', i') , where $A' \in \text{Ob}(\mathcal{C})$ and $i' \in \text{Mor}_{\mathcal{C}}(A', A)$ is injective. Given two subobjects (A', i') , (A'', i'') of A , (A', i') is said to be bigger than or equal to (A'', i'') (written $A' \geq A''$) if there exists a morphism $u: A'' \rightarrow A'$ in \mathcal{C} making the diagram

$$\begin{array}{ccc} A'' & \xrightarrow{u} & A' \\ i'' \searrow & & \swarrow i' \\ & A & \end{array}$$

commutative. In this case, u is injective and unique. For subobjects (A', i') , (A'', i'') and (A''', i''') of A , we have that $A' \geq A''$ and $A' \geq A'''$ if $A' \geq A''$ and $A'' \geq A'''$. Moreover, if $A' \geq A''$ and $A'' \geq A'$, with $i'' = i'u$ and $i' = i''v$ as above, then u and v are isomorphisms and $u^{-1} = v$.

A *quotient* of A is a subobject of A , A regarded as an object in the dual category \mathcal{C}^o . Therefore a quotient of A is a pair (Q', p') , where $Q' \in \text{Ob}(\mathcal{C})$ and $p' \in \text{Mor}(A, Q')$ is surjective. If (Q', p') , (Q'', p'') are quotients of A , (Q', p') is bigger than or equal to (Q'', p'') (written $Q' \geq Q''$) if there exists a morphism $w: Q' \rightarrow Q''$ in \mathcal{C} making the diagram

$$\begin{array}{ccc} Q' & \xrightarrow{w} & Q'' \\ p' \searrow & & \nearrow p'' \\ & A & \end{array}$$

commutative, w being surjective and unique.

Definition 2.18. Let \mathcal{C} be a category and $A, B \in \text{Ob}(\mathcal{C})$. Let $D \in \text{Ob}(\mathcal{C})$, $p_A \in \text{Mor}_{\mathcal{C}}(D, A)$ and $p_B \in \text{Mor}_{\mathcal{C}}(D, B)$. If, for each $C \in \text{Ob}(\mathcal{C})$, the mapping

$$u \in \text{Mor}_{\mathcal{C}}(C, D) \longmapsto (p_A u, p_B u) \in \text{Mor}_{\mathcal{C}}(C, A) \times \text{Mor}_{\mathcal{C}}(C, B)$$

is bijective, D is said to be a *product* of A, B through p_A, p_B .

Two products of A, B , if they do exist, are isomorphic. For this reason, we shall refer to the product of A, B , denoted by $A \times B$. We shall say that *finite products exist* in \mathcal{C} if, for all $A, B \in \text{Ob}(\mathcal{C})$, $A \times B$ exists.

Finite products exist in all the categories we have mentioned, as we shall now see (more generally, it can be proved [25] that products exist in all the categories considered in Examples 2.1 - 2.7 and in Examples 2.9 - 2.13).

Example 2.19. Finite products exist in Set .

In fact, let $A, B \in \text{Ob}(\text{Set})$ be arbitrary. If $A \times B$ is the product set and $p_A: A \times B \rightarrow A$ and $p_B: A \times B \rightarrow B$ are the projections, it is obvious that, for each $C \in \text{Ob}(\text{Set})$, the mapping

$$u \in \text{Mor}_{\text{Set}}(C, A \times B) \longmapsto (p_A \circ u, p_B \circ u) \in \text{Mor}_{\text{Set}}(C, A) \times \text{Mor}_{\text{Set}}(C, B)$$

is bijective.

Example 2.20. Finite products exist in Top .

In fact, let $A, B \in \text{Ob}(\text{Top})$ be arbitrary and let $A \times B, p_A, p_B$ be as in Example 2.19. If we endow $A \times B$ with the product topology, it follows from Example 2.19 and Proposition 4, p.28 of [6] that $A \times B$ is the product of A, B through the continuous mappings p_A, p_B .

Example 2.21. Finite products exist in Grp .

In fact, let $A, B \in \text{Ob}(\text{Grp})$ be arbitrary and let $A \times B, p_A, p_B$ be as in Example 2.19. If we consider $A \times B$ endowed with its group structure, it follows from Example 2.19 that $A \times B$ is the product of A, B through the group homomorphisms p_A, p_B .

By arguing as in Example 2.21, we have:

Example 2.22. Finite products exist in Mod_R .

Example 2.23. Finite products exist in Topm_R .

In fact, let $A, B \in \text{Ob}(\text{Topm}_R)$ be arbitrary and let $A \times B, p_A, p_B$ be as in Example 2.19. If we consider $A \times B$ endowed with its R -module structure and with the product topology, then $A \times B \in \text{Ob}(\text{Topm}_R)$ by Corollary 12.6 of [34]. Therefore, in view of Examples 2.20 and 2.22, $A \times B$ is the product of A, B through the continuous R -linear mappings p_A, p_B .

Example 2.24. Finite products exist in Ltm_R .

It suffices to argue as in Example 2.23, Corollary 12.6 of [34] being replaced by a remark on p.323 of [34].

By arguing as in Example 2.23, we get:

Example 2.25. Finite products exist in Ahtg .

Example 2.26. Finite products exist in Ban .

In fact, let $A, B \in \text{Ob}(\text{Ban})$ be arbitrary and let $A \times B, p_A, p_B$ be as in Example 2.19. If we consider $A \times B$ endowed with its vector space structure and with the product norm, then $A \times B \in \text{Ob}(\text{Ban})$ and it is easily verified that $A \times B$ is the product of A, B through the continuous linear mappings p_A, p_B .

By arguing as in Example 2.23, with the pertinent modifications, we get:

Example 2.27. Finite products exist in \mathbf{Lcs} .

Example 2.28. Finite products exist in \mathbf{Lcs}_K .

Example 2.29. Finite products exist in \mathbf{Borm}_R .

In fact, let $A, B \in \mathbf{Ob}(\mathbf{Borm}_R)$ be arbitrary and let $A \times B, p_A, p_B$ be as in Example 2.19. If we consider $A \times B$ endowed with its R -module structure and with the product bornology, it follows from Corollary 3 of [24] that $A \times B \in \mathbf{Ob}(\mathbf{Borm}_R)$ and is the product of A, B through the bounded R -linear mappings p_A, p_B .

Example 2.30. Finite products exist in \mathbf{Cbvs} .

In fact, let $A, B \in \mathbf{Ob}(\mathbf{Cbvs})$ and let $A \times B, p_A, p_B$ be as in Example 2.19. If we consider $A \times B$ endowed with its vector space structure and with the product bornology, it follows from Remark (1), p.31 of [16] that $A \times B \in \mathbf{Ob}(\mathbf{Cbvs})$ and is the product of A, B through the bounded linear mappings p_A, p_B .

By arguing as in Example 2.30, we get:

Example 2.31. Finite products exist in \mathbf{Cbvs}_K .

The dual notion to that considered in Definition 2.18 reads:

Definition 2.32. Let \mathcal{C} be a category and $A, B \in \mathbf{Ob}(\mathcal{C})$. Let $E \in \mathbf{Ob}(\mathcal{C})$, $q_A \in \mathbf{Mor}_{\mathcal{C}}(A, E)$ and $q_B \in \mathbf{Mor}_{\mathcal{C}}(B, E)$. If, for each $C \in \mathbf{Ob}(\mathcal{C})$, the mapping

$$u \in \mathbf{Mor}_{\mathcal{C}}(E, C) \longmapsto (uq_A, uq_B) \in \mathbf{Mor}_{\mathcal{C}}(A, C) \times \mathbf{Mor}_{\mathcal{C}}(B, C)$$

is bijective, E is said to be a *coproduct* of A, B through q_A, q_B .

Two coproducts of A, B , if they do exist, are isomorphic. For this reason, we shall refer to the coproduct of A, B , denoted by $A \sqcup B$. We shall say that *finite coproducts exist* in \mathcal{C} if, for all $A, B \in \mathbf{Ob}(\mathcal{C})$, $A \sqcup B$ exists.

Example 2.33. Finite coproducts exist in \mathbf{Set} .

In fact, let $A, B \in \mathbf{Ob}(\mathbf{Set})$ be arbitrary. Let $A \sqcup B$ be the sum of the sets A, B and $q_A: A \rightarrow A \sqcup B, q_B: B \rightarrow A \sqcup B$ the canonical mappings ([10], p.12). Then it is easily seen that, for each $C \in \mathbf{Ob}(\mathbf{Set})$, the mapping

$$u \in \mathbf{Mor}_{\mathbf{Set}}(A \sqcup B, C) \longmapsto (u \circ q_A, u \circ q_B) \in \mathbf{Mor}_{\mathbf{Set}}(A, C) \times \mathbf{Mor}_{\mathbf{Set}}(B, C)$$

is bijective.

Example 2.34. Finite coproducts exist in \mathbf{Top} .

In fact, let $A, B \in \mathbf{Ob}(\mathbf{Top})$ be arbitrary and let $A \sqcup B$, q_A , q_B be as in Example 2.33. If we consider $A \sqcup B$ endowed with the final topology for the mappings q_A , q_B , it follows from Example 2.33 and Proposition 6, p.31 of [6] that $A \sqcup B$ is the coproduct of A, B through the continuous mappings q_A , q_B .

Example 2.35. Finite coproducts exist in \mathbf{Grp} .

In fact, let $A, B \in \mathbf{Ob}(\mathbf{Grp})$ be arbitrary. Let $A \sqcup B$ be the free product of A, B and $q_A: A \rightarrow A \sqcup B$, $q_B: B \rightarrow A \sqcup B$ the canonical group homomorphisms. Then, by Proposition 12.3 of [20], $A \sqcup B$ is the coproduct of A, B through q_A , q_B .

In the sequel we shall see that finite coproducts exist in the categories \mathbf{Mod}_R , \mathbf{Topm}_R , \mathbf{Ltm}_R , \mathbf{Ahtg} , \mathbf{Ban} , \mathbf{Lcs} , \mathbf{Lcs}_K , \mathbf{Borm}_R , \mathbf{Cbvs} , \mathbf{Cbvs}_K . More generally, it can be proved [25] that coproducts exist in all the categories considered in Examples 2.1 - 2.6 and in Examples 2.9 - 2.13.

§3. Preliminaries on preadditive categories and examples

This section, strongly based on [2] and [14], is devoted to known facts about preadditive categories which will be needed in the sequel.

Definition 3.1. A category \mathcal{C} is said to be *preadditive* if the following axioms are satisfied:

- (a) for all $A, B \in \mathbf{Ob}(\mathcal{C})$, $\mathbf{Mor}_{\mathcal{C}}(A, B)$ is endowed with an abelian group structure (the identity element of $\mathbf{Mor}_{\mathcal{C}}(A, B)$ will be denoted by 0_{AB});
- (b) for all $A, B, C \in \mathbf{Ob}(\mathcal{C})$, the composition of morphisms

$$\mathbf{Mor}_{\mathcal{C}}(A, B) \times \mathbf{Mor}_{\mathcal{C}}(B, C) \longrightarrow \mathbf{Mor}_{\mathcal{C}}(A, C)$$

is a \mathbb{Z} -bilinear mapping;

- (c) there exists an $A \in \mathbf{Ob}(\mathcal{C})$ such that $1_A = 0_{AA}$.

Axiom (c) does not follow from the other axioms of a preadditive category: for instance, consider the category of non-trivial abelian groups.

It is obvious that the categories \mathbf{Set} , \mathbf{Top} and \mathbf{Grp} are not preadditive, and that the categories \mathbf{Mod}_R , \mathbf{Topm}_R , \mathbf{Ltm}_R , \mathbf{Ahtg} , \mathbf{Ban} , \mathbf{Lcs} , \mathbf{Lcs}_K , \mathbf{Borm}_R , \mathbf{Cbvs} and \mathbf{Cbvs}_K are preadditive.

Remark 3.2. Let \mathcal{C} be a preadditive category and $A, B \in \mathbf{Ob}(\mathcal{C})$. It can be shown that $A \times B$ exists if and only if $A \sqcup B$ exists; in this case, $A \times B$ and $A \sqcup B$ are isomorphic.

Consequently, by Examples 22-31, finite coproducts exist in the categories \mathbf{Mod}_R , \mathbf{Topm}_R , \mathbf{Ltm}_R , \mathbf{Ahtg} , \mathbf{Ban} , \mathbf{Lcs} , \mathbf{Lcs}_K , \mathbf{Borm}_R , \mathbf{Cbvs} and \mathbf{Cbvs}_K .

Remark 3.3. Let \mathcal{C} be a preadditive category and $u \in \mathbf{Mor}_{\mathcal{C}}(A, B)$. For each $C \in \mathbf{Ob}(\mathcal{C})$ let u^* (resp. u_*) be the group homomorphism from $\mathbf{Mor}_{\mathcal{C}}(C, A)$ into $\mathbf{Mor}_{\mathcal{C}}(C, B)$ (resp. from $\mathbf{Mor}_{\mathcal{C}}(B, C)$ into $\mathbf{Mor}_{\mathcal{C}}(A, C)$) given by $u^*(v) = uv$ for $v \in \mathbf{Mor}_{\mathcal{C}}(C, A)$ (resp. $u_*(w) = wu$ for $w \in \mathbf{Mor}_{\mathcal{C}}(B, C)$). It is clear that u is injective (resp. surjective) if and only if the sequence

$$0 \longrightarrow \mathbf{Mor}_{\mathcal{C}}(C, A) \xrightarrow{u^*} \mathbf{Mor}_{\mathcal{C}}(C, B) \quad (\text{resp. } 0 \longrightarrow \mathbf{Mor}_{\mathcal{C}}(B, C) \xrightarrow{u_*} \mathbf{Mor}_{\mathcal{C}}(A, C))$$

of group homomorphisms is exact for each $C \in \mathbf{Ob}(\mathcal{C})$ ([20], p.15).

Definition 3.4. Let \mathcal{C} be a preadditive category. An object A in \mathcal{C} as in Definition 3.1(c) is said to be a *zero* of \mathcal{C} .

For $A \in \mathbf{Ob}(\mathcal{C})$, the following conditions are equivalent:

- (a) A is a zero of \mathcal{C} ;
- (b) $\mathbf{Mor}_{\mathcal{C}}(A, A) = \{0_{AA}\}$;
- (c) $\mathbf{Mor}_{\mathcal{C}}(A, C) = \{0_{AC}\}$ for all $C \in \mathbf{Ob}(\mathcal{C})$;
- (d) $\mathbf{Mor}_{\mathcal{C}}(C, A) = \{0_{CA}\}$ for all $C \in \mathbf{Ob}(\mathcal{C})$.

Consequently, between two zeros of \mathcal{C} there is only one isomorphism, namely, the unique morphism between them. For this reason, we shall refer to the zero of \mathcal{C} , denoted by 0 .

Proposition 3.5. Let \mathcal{C} be a preadditive category and $u \in \mathbf{Mor}_{\mathcal{C}}(A, B)$. For $L \in \mathbf{Ob}(\mathcal{C})$ and $i \in \mathbf{Mor}_{\mathcal{C}}(L, A)$, the following conditions are equivalent:

$$(a) \quad 0 \longrightarrow \text{Mor}_{\mathcal{C}}(C, L) \xrightarrow{i^*} \text{Mor}_{\mathcal{C}}(C, A) \xrightarrow{u^*} \text{Mor}_{\mathcal{C}}(C, B)$$

is an exact sequence of group homomorphisms for each $C \in \text{Ob}(\mathcal{C})$.

(b) (1) i is injective;

(2) $ui = 0_{LB}$;

(3) for all $C \in \text{Ob}(\mathcal{C})$ and for all $v \in \text{Mor}_{\mathcal{C}}(C, A)$ such that $uv = 0_{CB}$, there exists a $w \in \text{Mor}_{\mathcal{C}}(C, L)$ such that $v = iw$.

Therefore, if (L, i) and (L', i') satisfy condition (a) of Proposition 3.5, then the subobjects (L, i) and (L', i') of A are such that $L \geq L'$ and $L' \geq L$, and so are isomorphic.

Definition 3.6. Let \mathcal{C} be a preadditive category and $u \in \text{Mor}_{\mathcal{C}}(A, B)$. A pair $(\text{Ker}(u), i)$, where $\text{Ker}(u) \in \text{Ob}(\mathcal{C})$ and $i \in \text{Mor}_{\mathcal{C}}(\text{Ker}(u), A)$, is said to be the *kernel* of u if the sequence

$$0 \longrightarrow \text{Mor}_{\mathcal{C}}(C, \text{Ker}(u)) \xrightarrow{i^*} \text{Mor}_{\mathcal{C}}(C, A) \xrightarrow{u^*} \text{Mor}_{\mathcal{C}}(C, B)$$

of group homomorphisms is exact for each $C \in \text{Ob}(\mathcal{C})$.

The subobject $(\text{Ker}(u), i)$ of A , if it exists, is essentially unique, as we have just observed.

Example 3.7. Let $u \in \text{Mor}_{\text{Mod}_R}(A, B)$ and $C \in \text{Ob}(\text{Mod}_R)$ be arbitrary. Put $\text{Ker}(u) = \{x \in A; u(x) = 0\}$ and let $i: \text{Ker}(u) \rightarrow A$ be the inclusion mapping. Obviously, $\text{Ker}(u) \in \text{Ob}(\text{Mod}_R)$ and $i \in \text{Mor}_{\text{Mod}_R}(\text{Ker}(u), A)$. Since i is an injective morphism in Mod_R , Remark 3.3 guarantees the exactness of the sequence

$$0 \longrightarrow \text{Mor}_{\text{Mod}_R}(C, \text{Ker}(u)) \xrightarrow{i^*} \text{Mor}_{\text{Mod}_R}(C, A).$$

Now, let us consider the sequence

$$\text{Mor}_{\text{Mod}_R}(C, \text{Ker}(u)) \xrightarrow{i^*} \text{Mor}_{\text{Mod}_R}(C, A) \xrightarrow{u^*} \text{Mor}_{\text{Mod}_R}(C, B).$$

If $w \in \text{Mor}_{\text{Mod}_R}(C, \text{Ker}(u))$,

$$u^*(i^*(w)) = u^*(i \circ w) = u \circ (i \circ w) = (u \circ i) \circ w = 0_{\text{Ker}(u)B} \circ w = 0_{CB};$$

thus $i^*(w) \in \text{Ker}(u^*)$ and $\text{Im}(i^*) \subset \text{Ker}(u^*)$. On the other hand, if $v \in \text{Ker}(u^*)$, $\{v(x); x \in C\} \subset \text{Ker}(u)$, and we can view v as an R -linear mapping w from C into $\text{Ker}(u)$.

Moreover, $i^*(w) = i \circ w = v$, and hence $\text{Ker}(u^*) \subset \text{Im}(i^*)$. Therefore $\text{Im}(i^*) = \text{Ker}(u^*)$, and we have proved that $(\text{Ker}(u), i)$ is the kernel of u .

Example 3.8. Let $u \in \text{Mor}_{\text{Topm}_R}(A, B)$ and $C \in \text{Ob}(\text{Topm}_R)$ be arbitrary. Let $\text{Ker}(u)$ and i be as in Example 3.7 and consider $\text{Ker}(u)$ endowed with the topology induced by that of A . Then $\text{Ker}(u) \in \text{Ob}(\text{Topm}_R)$ ([34], p.87) and $i \in \text{Mor}_{\text{Topm}_R}(\text{Ker}(u), A)$. We claim that $(\text{Ker}(u), i)$ is the kernel of u . Indeed, since i is an injective morphism in Topm_R , Remark 3.3 guarantees the exactness of the sequence

$$0 \longrightarrow \text{Mor}_{\text{Topm}_R}(C, \text{Ker}(u)) \xrightarrow{i^*} \text{Mor}_{\text{Topm}_R}(C, A).$$

Now, let us consider the sequence

$$\text{Mor}_{\text{Topm}_R}(C, \text{Ker}(u)) \xrightarrow{i^*} \text{Mor}_{\text{Topm}_R}(C, A) \xrightarrow{u^*} \text{Mor}_{\text{Topm}_R}(C, B).$$

As in Example 3.7, $\text{Im}(i^*) \subset \text{Ker}(u^*)$. On the other hand, if $v \in \text{Ker}(u^*)$, consider w as in Example 3.7. Then it is clear that $w \in \text{Mor}_{\text{Topm}_R}(C, \text{Ker}(u))$ and $i^*(w) = v$; thus $\text{Ker}(u^*) \subset \text{Im}(i^*)$. Therefore $\text{Im}(i^*) = \text{Ker}(u^*)$ and our claim is justified.

By arguing exactly as in Example 3.8 (see a remark on p.323 of [34]), one concludes:

Example 3.9. Every morphism in Ltm_R has a kernel.

Example 3.10. Let $u \in \text{Mor}_{\text{Ahtg}}(A, B)$ be arbitrary and let $\text{Ker}(u)$ and i be as in Example 3.8. Then $\text{Ker}(u) \in \text{Ob}(\text{Ahtg})$, $i \in \text{Mor}_{\text{Ahtg}}(\text{Ker}(u), A)$ and, by arguing exactly as in Example 3.8, one shows that $(\text{Ker}(u), i)$ is the kernel of u .

Example 3.11. Let $u \in \text{Mor}_{\text{Ban}}(A, B)$ be arbitrary and let $\text{Ker}(u)$ and i be as in Example 3.8. Since $\text{Ker}(u)$ is a closed subspace of A , $\text{Ker}(u) \in \text{Ob}(\text{Ban})$ under the norm induced by that of A . Finally, $i \in \text{Mor}_{\text{Ban}}(\text{Ker}(u), A)$ and, by arguing as in Example 3.8, one shows that $(\text{Ker}(u), i)$ is the kernel of u .

By arguing exactly as in Example 3.8, one justifies the validity of the two assertions below:

Example 3.12. Every morphism in Lcs has a kernel.

Example 3.13. Every morphism in Lcs_K has a kernel.

Example 3.14. Let $u \in \text{Mor}_{\text{Borm}_R}(A, B)$ be arbitrary and let $\text{Ker}(u)$ and i be as in Example 3.7. Consider $\text{Ker}(u)$ endowed with the bornology induced by that of A ; then $\text{Ker}(u) \in \text{Ob}(\text{Borm}_R)$ ([24], Corollary 1.a)) and $i \in \text{Mor}_{\text{Borm}_R}(\text{Ker}(u), A)$. We claim that $(\text{Ker}(u), i)$ is the kernel of u . Indeed, let $C \in \text{Ob}(\text{Borm}_R)$ be arbitrary. Since i is an injective morphism in Borm_R , Remark 3.3 furnishes the exactness of the sequence

$$0 \longrightarrow \text{Mor}_{\text{Borm}_R}(C, \text{Ker}(u)) \xrightarrow{i^*} \text{Mor}_{\text{Borm}_R}(C, A).$$

Now, let us consider the sequence

$$\text{Mor}_{\text{Borm}_R}(C, \text{Ker}(u)) \xrightarrow{i^*} \text{Mor}_{\text{Borm}_R}(C, A) \xrightarrow{u^*} \text{Mor}_{\text{Borm}_R}(C, B).$$

As in Example 3.7, $\text{Im}(i^*) \subset \text{Ker}(u^*)$. On the other hand, if $v \in \text{Ker}(u^*)$, take w as in Example 3.7. Then it is clear that $w \in \text{Mor}_{\text{Borm}_R}(C, \text{Ker}(u))$ and $i^*(w) = v$. Hence $\text{Ker}(u^*) \subset \text{Im}(i^*)$, and $\text{Im}(i^*) = \text{Ker}(u^*)$. Therefore $(\text{Ker}(u), i)$ is the kernel of u .

By arguing exactly as in Example 3.14, one justifies the validity of the two assertions below:

Example 3.15. Every morphism in Cbvs has a kernel.

Example 3.16. Every morphism in Cbvs_K has a kernel.

By duality (recall Proposition 3.5), the following result holds:

Proposition 3.17. *Let \mathcal{C} be a preadditive category and $u \in \text{Mor}_{\mathcal{C}}(A, B)$. For $J \in \text{Ob}(\mathcal{C})$ and $j \in \text{Mor}_{\mathcal{C}}(B, J)$, the following conditions are equivalent:*

$$(a) \quad 0 \longrightarrow \text{Mor}_{\mathcal{C}}(J, C) \xrightarrow{j^*} \text{Mor}_{\mathcal{C}}(B, C) \xrightarrow{u_*} \text{Mor}_{\mathcal{C}}(A, C)$$

is an exact sequence of group homomorphisms for each $C \in \text{Ob}(\mathcal{C})$.

(b) (1) j is surjective;

(2) $ju = 0_{AJ}$;

(3) for all $C \in \text{Ob}(\mathcal{C})$ and for all $w \in \text{Mor}_{\mathcal{C}}(B, C)$ such that $wu = 0_{AC}$, there exists a $v \in \text{Mor}_{\mathcal{C}}(J, C)$ such that $w = vj$.

Therefore, if (J, j) and (J', j') satisfy condition (a) of Proposition 3.17, then the quotients (J, j) and (J', j') of B are such that $J \geq J'$ and $J' \geq J$, and so are isomorphic.

Definition 3.18. Let \mathcal{C} be a preadditive category and $u \in \text{Mor}_{\mathcal{C}}(A, B)$. A pair $(\text{Coker}(u), j)$, where $\text{Coker}(u) \in \text{Ob}(\mathcal{C})$ and $j \in \text{Mor}_{\mathcal{C}}(B, \text{Coker}(u))$, is said to be the *cokernel* of u if the sequence

$$0 \longrightarrow \text{Mor}_{\mathcal{C}}(\text{Coker}(u), C) \xrightarrow{j_*} \text{Mor}_{\mathcal{C}}(B, C) \xrightarrow{u_*} \text{Mor}_{\mathcal{C}}(A, C)$$

of group homomorphisms is exact for each $C \in \text{Ob}(\mathcal{C})$.

The quotient $(\text{Coker}(u), j)$ of B , if it exists, is essentially unique, as we have just observed.

Example 3.19. Let $u \in \text{Mor}_{\text{Mod}_R}(A, B)$ be arbitrary and let M be the submodule $\{u(x); x \in A\}$ of B . Put $\text{Coker}(u) = B/M$ (regarded as an R -module) and let $\pi: B \rightarrow \text{Coker}(u)$ be the canonical R -linear mapping. We claim that $(\text{Coker}(u), \pi)$ is the cokernel of u . Indeed, let $C \in \text{Ob}(\text{Mod}_R)$ be arbitrary. Since π is a surjective morphism in Mod_R , the sequence

$$0 \longrightarrow \text{Mor}_{\text{Mod}_R}(\text{Coker}(u), C) \xrightarrow{\pi_*} \text{Mor}_{\text{Mod}_R}(B, C)$$

is exact by Remark 3.3. Now, let us consider the sequence

$$\text{Mor}_{\text{Mod}_R}(\text{Coker}(u), C) \xrightarrow{\pi_*} \text{Mor}_{\text{Mod}_R}(B, C) \xrightarrow{u_*} \text{Mor}_{\text{Mod}_R}(A, C).$$

If $v \in \text{Mor}_{\mathcal{C}}(\text{Coker}(u), C)$,

$$u_*(\pi_*(v)) = u_*(v \circ \pi) = (v \circ \pi) \circ u = v \circ (\pi \circ u) = v \circ 0_{A\text{Coker}(u)} = 0_{AC};$$

hence $\text{Im}(\pi_*) \subset \text{Ker}(u_*)$. On the other hand, let $w \in \text{Ker}(u_*)$ and define $v: \text{Coker}(u) \rightarrow C$ by $v(\pi(y)) = w(y)$ for $y \in B$ (v is well defined since $\text{Ker}(\pi) \subset \text{Ker}(w)$). Then $v \in \text{Mor}_{\text{Mod}_R}(\text{Coker}(u), C)$ and $w = v \circ \pi = \pi_*(v)$; thus $\text{Ker}(u_*) \subset \text{Im}(\pi_*)$. Therefore $\text{Im}(\pi_*) = \text{Ker}(u_*)$, and $(\text{Coker}(u), \pi)$ is the cokernel of u .

Example 3.20. Let $u \in \text{Mor}_{\text{Topm}_R}(A, B)$ be arbitrary and let $\text{Coker}(u)$ and π be as in Example 3.19. By considering $\text{Coker}(u)$ endowed with the quotient topology, it follows that $\text{Coker}(u) \in \text{Ob}(\text{Topm}_R)$ ([34], Theorem 12.10) and $\pi \in \text{Mor}_{\text{Topm}_R}(B, \text{Coker}(u))$. We claim that $(\text{Coker}(u), \pi)$ is the cokernel of u . Indeed, let $C \in \text{Ob}(\text{Topm}_R)$ be arbitrary. Since π is a surjective morphism in Topm_R , the sequence

$$0 \longrightarrow \text{Mor}_{\text{Topm}_R}(\text{Coker}(u), C) \xrightarrow{\pi_*} \text{Mor}_{\text{Topm}_R}(B, C)$$

is exact by Remark 3.3. Now, let us consider the sequence

$$\mathrm{Mor}_{\mathrm{Topm}_R}(\mathrm{Coker}(u), C) \xrightarrow{\pi_*} \mathrm{Mor}_{\mathrm{Topm}_R}(B, C) \xrightarrow{u_*} \mathrm{Mor}_{\mathrm{Topm}_R}(A, C).$$

As in Example 3.19, $\mathrm{Im}(\pi_*) \subset \mathrm{Ker}(u_*)$. On the other hand, if $w \in \mathrm{Ker}(u_*)$, define v as in Example 3.19. Then $v \in \mathrm{Mor}_{\mathrm{Topm}_R}(\mathrm{Coker}(u), C)$. In fact, if V is a neighborhood of 0 in C there is a neighborhood U of 0 in B such that $w(U) \subset V$. Since $v(\pi(U)) = w(U)$, $\pi(U)$ being a neighborhood of 0 in $\mathrm{Coker}(u)$ because π is open, the continuity of v is verified. Finally, since $w = \pi_*(v)$, the inclusion $\mathrm{Ker}(u_*) \subset \mathrm{Im}(\pi_*)$ is valid. Therefore $\mathrm{Im}(\pi_*) = \mathrm{Ker}(u_*)$, and our claim is proved.

Example 3.21. Let $u \in \mathrm{Mor}_{\mathrm{Ltm}_R}(A, B)$ be arbitrary and let $\mathrm{Coker}(u)$ and π be as in Example 3.20. Then it is clear that $\mathrm{Coker}(u) \in \mathrm{Ob}(\mathrm{Ltm}_R)$ and $\pi \in \mathrm{Mor}_{\mathrm{Ltm}_R}(B, \mathrm{Coker}(u))$. Moreover, by arguing exactly as in Example 3.20, one shows that $(\mathrm{Coker}(u), \pi)$ is the cokernel of u .

Example 3.22. Let $u \in \mathrm{Mor}_{\mathrm{Ahtg}}(A, B)$ be arbitrary. Then the set $M = \{\overline{u(x)}; x \in A\}$ is a closed subgroup of B , and hence the abelian group $\mathrm{Coker}(u) = B/M$, endowed with the quotient topology, is an object in Ahtg . If $\pi: B \rightarrow \mathrm{Coker}(u)$ denotes the canonical group homomorphism, which is a morphism in Ahtg , the argument used in Example 3.20 shows that $(\mathrm{Coker}(u), \pi)$ is the cokernel of u .

Example 3.23. Let $u \in \mathrm{Mor}_{\mathrm{Ban}}(A, B)$ be arbitrary. Then the set $M = \{\overline{u(x)}; x \in A\}$ is a closed subspace of B , and hence the vector space $\mathrm{Coker}(u) = B/M$, endowed with the quotient norm, is an object in Ban . If $\pi: B \rightarrow \mathrm{Coker}(u)$ denotes the canonical linear mapping, which is a morphism in Ban , the argument used in Example 3.20 shows that $(\mathrm{Coker}(u), \pi)$ is the cokernel of u .

By arguing as in Example 3.20, one shows the validity of the two assertions below:

Example 3.24. Every morphism in Lcs admits a cokernel.

Example 3.25. Every morphism in Lcs_K admits a cokernel.

Example 3.26. Let $u \in \mathrm{Mor}_{\mathrm{Born}_R}(A, B)$ be arbitrary and let $\mathrm{Coker}(u)$ and π be as in Example 3.19. By considering $\mathrm{Coker}(u)$ endowed with the quotient bornology, we have that

$\text{Coker}(u) \in \text{Ob}(\text{Borm}_R)$ and $\pi \in \text{Mor}_{\text{Borm}_R}(B, \text{Coker}(u))$. We claim that $(\text{Coker}(u), \pi)$ is the cokernel of u . Indeed, let $C \in \text{Ob}(\text{Borm}_R)$ be arbitrary. Since π is a surjective morphism in Borm_R , Remark 3.3 implies the exactness of the sequence

$$0 \longrightarrow \text{Mor}_{\text{Borm}_R}(\text{Coker}(u), C) \xrightarrow{\pi_*} \text{Mor}_{\text{Borm}_R}(B, C).$$

Now, let us show the exactness of the sequence

$$\text{Mor}_{\text{Borm}_R}(\text{Coker}(u), C) \xrightarrow{\pi_*} \text{Mor}_{\text{Borm}_R}(B, C) \xrightarrow{u_*} \text{Mor}_{\text{Borm}_R}(A, C).$$

As in Example 3.19, $\text{Im}(\pi_*) \subset \text{Ker}(u_*)$. On the other hand, if $w \in \text{Ker}(u_*)$, define v as in Example 3.19. Then $v \in \text{Mor}_{\text{Borm}_R}(\text{Coker}(u), C)$ because w is bounded and $v(\pi(L)) = w(L)$ for every bounded subset L of B . Finally, since $w = \pi_*(v)$, the inclusion $\text{Ker}(u_*) \subset \text{Im}(\pi_*)$ is valid. Therefore $\text{Im}(\pi_*) = \text{Ker}(u_*)$, and our claim is proved.

By arguing as in the preceding example, one justifies the validity of the two assertions below:

Example 3.27. Every morphism in Cbvs has a cokernel.

Example 3.28. Every morphism in Cbvs_K has a cokernel.

Definition 3.29. Let \mathcal{C} be a preadditive category satisfying the following axiom:

(d) every morphism in \mathcal{C} admits kernel and cokernel.

For $u \in \text{Mor}_{\mathcal{C}}(A, B)$, we define the *image* of u , denoted by $\text{Im}(u)$, as $\text{Im}(u) = \text{Ker}(\text{Coker}(u))$, and the *coimage* of u , denoted by $\text{Coim}(u)$, as $\text{Coim}(u) = \text{Coker}(\text{Ker}(u))$

$$\begin{array}{ccccc} \text{Ker}(u) & \xrightarrow{i} & A & \xrightarrow{j'} & \text{Coim}(u) \\ & & \downarrow u & & \\ \text{Im}(u) & \xrightarrow{i'} & B & \xrightarrow{j} & \text{Coker}(u) \end{array}$$

As we have already seen, the preadditive categories Mod_R , Topm_R , Ltm_R , Ahtg , Ban , Lcs , Lcs_K , Borm_R , Cbvs and Cbvs_K satisfy axiom (d).

In the statement of the next result, the equality $\text{Coim}(u) = A$ (resp. $\text{Im}(u) = B$) will mean that the morphism $j': A \rightarrow \text{Coim}(u)$ (resp. $i': \text{Im}(u) \rightarrow B$) is an isomorphism.

Proposition 3.30. *Let \mathcal{C} be a category as in Definition 3.29 and let $u \in \text{Mor}_{\mathcal{C}}(A, B)$. Then the following assertions hold:*

- (a) *u is injective if and only if $\text{Ker}(u) = 0$;*
- (b) *u is injective if and only if $\text{Coim}(u) = A$;*
- (c) *u is surjective if and only if $\text{Coker}(u) = 0$;*
- (d) *u is surjective if and only if $\text{Im}(u) = B$.*

The following basic result, whose proof we shall include here, will be important for our purposes.

Proposition 3.31. *Let \mathcal{C} be a category as in Definition 3.29 and let $u \in \text{Mor}_{\mathcal{C}}(A, B)$. Then there exists a unique $\bar{u} \in \text{Mor}_{\mathcal{C}}(\text{Coim}(u), \text{Im}(u))$ such that $u = i' \bar{u} j'$, i' and j' being as in Definition 3.29.*

$$A \xrightarrow{j'} \text{Coim}(u) \xrightarrow{\bar{u}} \text{Im}(u) \xrightarrow{i'} B.$$

Proof. In order to prove the uniqueness, assume that $u = i' \bar{u} j'$ and $u = i' \bar{u}' j'$, where $\bar{u}, \bar{u}' \in \text{Mor}_{\mathcal{C}}(\text{Coim}(u), \text{Im}(u))$. Since j' is surjective and $(i' \bar{u}) j' = (i' \bar{u}') j'$, then $i' \bar{u} = i' \bar{u}'$. Thus $\bar{u} = \bar{u}'$ because i' is injective.

Now, let us prove the existence. Indeed, let $(\text{Ker}(u), i)$ be the kernel of u and $(\text{Coker}(u), j)$ the cokernel of u . Since $ui = 0_{\text{Ker}(u)B}$ by Proposition 3.5, there is a morphism $v: \text{Coim}(u) \rightarrow B$ in \mathcal{C} such that $u = v j'$ (Proposition 3.17). On the other hand, since $ju = 0_{A \text{Coker}(u)}$ by Proposition 3.17, then

$$(jv) j' = j(v j') = ju = 0_{A \text{Coker}(u)} = 0_{\text{Coim}(u) \text{Coker}(u)} j'.$$

Thus, by the surjectivity of j' , $jv = 0_{\text{Coim}(u) \text{Coker}(u)}$. Therefore, by Proposition 3.5, there is a morphism $\bar{u}: \text{Coim}(u) \rightarrow \text{Im}(u)$ in \mathcal{C} such that $v = i' \bar{u}$. Consequently, $u = v j' = i' \bar{u} j'$, which concludes the proof.

The proofs of the next results may be found in [2].

Proposition 3.32. *Let \mathcal{C} be a category as in Definition 3.29, $u \in \text{Mor}_{\mathcal{C}}(A, B)$ and $v \in \text{Mor}_{\mathcal{C}}(B, C)$. Then the following assertions hold:*

- (a) *$\text{Ker}(vu) \geq \text{Ker}(u)$, and $\text{Ker}(vu) \leq \text{Ker}(u)$ if v is injective;*
- (b) *$\text{Coker}(vu) \geq \text{Coker}(v)$, and $\text{Coker}(vu) \leq \text{Coker}(v)$ if u is surjective.*

Corollary 3.33. *For \mathcal{C} , u and v as in Proposition 3.32, the following assertions hold:*

- (a) $\text{Coim}(vu) \leq \text{Coim}(u)$, and $\text{Coim}(vu) \geq \text{Coim}(u)$ if v is injective;
- (b) $\text{Im}(vu) \leq \text{Im}(v)$, and $\text{Im}(vu) \geq \text{Im}(v)$ if u is surjective.

Corollary 3.34. *Let \mathcal{C} be a category as in Definition 3.29 and $A \in \text{Ob}(\mathcal{C})$. Then the following assertions hold:*

- (a) if (A', i') and (A'', i'') are subobjects of A such that $A' \leq A''$, then $\text{Coker}(i') \geq \text{Coker}(i'')$;
- (b) if (Q', p') and (Q'', p'') are quotients of A such that $Q' \leq Q''$, then $\text{Ker}(p') \geq \text{Ker}(p'')$.

Proof. (a): Since $A' \leq A''$, there is a morphism $u: A' \rightarrow A''$ in \mathcal{C} such that $i' = i''u$. Thus, by Proposition 3.32(b), $\text{Coker}(i') \geq \text{Coker}(i'')$.

(b): Follows from (a), by duality.

§4. Semiabelian categories

Definition 4.1 [22]. A category \mathcal{C} is said to be *semiabelian* if it is preadditive, satisfies axiom (d), as well as the following axioms:

- (e) finite products exist in \mathcal{C} ;
- (f) for all $A, B \in \text{Ob}(\mathcal{C})$ and for all $u \in \text{Mor}_{\mathcal{C}}(A, B)$, the morphism $\bar{u}: \text{Coim}(u) \rightarrow \text{Im}(u)$ (Proposition 3.31) is bijective.

By Remark 3.2, finite coproducts exist in any semiabelian category.

By what we have seen in §2 and §3, the categories Mod_R , Topm_R , Ltm_R , Ahtg , Ban , Lcs , Lcs_K , Borm_R , Cbvs and Cbvs_K are preadditive and satisfy axioms (d) and (e). In the next examples we shall see that they also satisfy axiom (f), and hence are semiabelian.

Example 4.2. Mod_R is a semiabelian category.

Indeed, let $u \in \text{Mor}_{\text{Mod}_R}(A, B)$ be arbitrary. Then $\text{Coim}(u) = \text{Coker}(\text{Ker}(u)) = A/\text{Ker}(u)$, $\text{Im}(u) = \text{Ker}(\text{Coker}(u)) = \{u(x); x \in A\}$ and $\bar{u}: \text{Coim}(u) \rightarrow \text{Im}(u)$ is the R -linear mapping given by $\bar{u}(x + \text{Ker}(u)) = u(x)$ for $x \in A$.

Now, let $C \in \text{Ob}(\text{Mod}_R)$ be arbitrary, and consider the sequences

$$0 \longrightarrow \text{Mor}_{\text{Mod}_R}(C, A/\text{Ker}(u)) \xrightarrow{(\bar{u})^*} \text{Mor}_{\text{Mod}_R}(C, \text{Im}(u))$$

and

$$0 \longrightarrow \text{Mor}_{\text{Mod}_R}(\text{Im}(u), C) \xrightarrow{(\bar{u})_*} \text{Mor}_{\text{Mod}_R}(A/\text{Ker}(u), C).$$

If $v \in \text{Mor}_{\text{Mod}_R}(C, A/\text{Ker}(u))$ and $(\bar{u})^*(v) = \bar{u} \circ v = 0_{C/\text{Im}(u)}$, then $v = 0_{C/\text{Ker}(u)}$. In fact, if $t \in C$ is arbitrary, there is an $x \in A$ with $v(t) = x + \text{Ker}(u)$; thus $\bar{u}(v(t)) = u(x) = 0$ (that is, $x \in \text{Ker}(u)$), and $v(t) = \text{Ker}(u)$. This shows the exactness of the first sequence. And, if $w \in \text{Mor}_{\text{Mod}_R}(\text{Im}(u), C)$ and $(\bar{u})_*(w) = w \circ \bar{u} = 0_{A/\text{Ker}(u)/C}$, then $w = 0_{\text{Im}(u)/C}$. In fact, if $x \in A$ is arbitrary,

$$0 = (w \circ \bar{u})(x + \text{Ker}(u)) = w(u(x)).$$

This shows the exactness of the second sequence. Therefore, in view of Remark 3.3, the morphism \bar{u} is bijective (more precisely, it is easily seen that \bar{u} is an isomorphism in Mod_R).

By the same procedure (and with the pertinent modifications), we conclude that the categories Topm_R , Ltm_R , Lcs , Lcs_K , Borm_R , Cbvs and Cbvs_K are semiabelian.

Example 4.3. Ahtg is a semiabelian category.

Indeed, let $u \in \text{Mor}_{\text{Ahtg}}(A, B)$ be arbitrary. Then $\text{Coim}(u) = \text{Coker}(\text{Ker}(u)) = A/\text{Ker}(u)$ (endowed with the quotient topology; recall Examples 3.10 and 3.22), and $\text{Im}(u) = \text{Ker}(\text{Coker}(u)) = \overline{\{u(x); x \in A\}}$ (endowed with the topology induced by that B). Let $C \in \text{Ob}(\text{Ahtg})$ be arbitrary, and consider the sequences

$$0 \longrightarrow \text{Mor}_{\text{Ahtg}}(C, A/\text{Ker}(u)) \xrightarrow{(\bar{u})^*} \text{Mor}_{\text{Ahtg}}(C, \text{Im}(u))$$

and

$$0 \longrightarrow \text{Mor}_{\text{Ahtg}}(\text{Im}(u), C) \xrightarrow{(\bar{u})_*} \text{Mor}_{\text{Ahtg}}(A/\text{Ker}(u), C),$$

where \bar{u} is the morphism in Ahtg given by $\bar{u}(x + \text{Ker}(u)) = u(x)$ for $x \in A$. The exactness of the first sequence follows exactly as in Example 4.2. Moreover, if $w \in \text{Mor}_{\text{Ahtg}}(\text{Im}(u), C)$ and $(\bar{u})_*(w) = 0_{A/\text{Ker}(u)/C}$, then $w = 0_{\text{Im}(u)/C}$. In fact, if $x \in A$ is arbitrary, $w(u(x)) = 0$

as we have seen above, and the continuity of w implies $w = 0_{\text{Im}(u)C}$. This shows the exactness of the second sequence. Therefore, in view of Remark 3.3, the morphism \bar{u} is bijective, and (f) holds.

Example 4.4. Ban is a semiabelian category.

Indeed, let $u \in \text{Mor}_{\text{Ban}}(A, B)$ be arbitrary. Then $\text{Coim}(u) = A/\text{Ker}(u)$ (endowed with the quotient norm), and $\text{Im}(u) = \{\overline{u(x)}; x \in A\}$ (endowed with the norm induced by that of B). By arguing exactly as in the preceding example, we conclude that the morphism \bar{u} is bijective, and (f) holds.

Proposition 4.5. *Let \mathcal{C} be a semiabelian category, $u \in \text{Mor}_{\mathcal{C}}(A, B)$ and $v \in \text{Mor}_{\mathcal{C}}(B, C)$. Then the following conditions are equivalent:*

- (a) $vu = 0_{AC}$;
- (b) $\text{Im}(u) \leq \text{Ker}(v)$;
- (c) $\text{Coim}(v) \leq \text{Coker}(u)$.

$$\begin{array}{ccccc} A & \xrightarrow{j} & \text{Coim}(u) & \xrightarrow{\bar{u}} & \text{Im}(u) & \xrightarrow{i} & B & \xrightarrow{v} & C \\ & & & & \uparrow i' & & & & \\ & & & & \text{Ker}(v) & & & & \end{array}$$

Proof. Since (b) and (c) are dual assertions, it is enough to prove that (a) and (b) are equivalent.

(a) \Rightarrow (b): Since $0_{AC} = vu = v(i\bar{u}j) = (vi\bar{u})j = 0_{\text{Coim}(u)C}j$ and since j is surjective, we have $(vi)\bar{u} = 0_{\text{Coim}(u)C} = 0_{\text{Im}(u)C}\bar{u}$. Thus, by the surjectivity of \bar{u} , $vi = 0_{\text{Im}(u)C}$, and Proposition 3.5 furnishes a morphism $w: \text{Im}(u) \rightarrow \text{Ker}(v)$ such that $i = i'w$; hence $\text{Im}(u) \leq \text{Ker}(v)$.

(b) \Rightarrow (a): By hypothesis we can write $i = i'w$, for w as above. Finally, the relations $vu = (vi)(\bar{u}j)$ and $vi = (vi')w = 0_{\text{Ker}(v)C}w = 0_{\text{Im}(u)C}$ give $vu = 0_{AC}$.

It seems that the notion of a strict morphism was first considered by Weil [35] in his celebrated memoir on topological groups. In our context, such a concept reads as follows:

Definition 4.6 [26]. Let \mathcal{C} be a semiabelian category. A morphism u in \mathcal{C} is said to be *strict* if the corresponding morphism \bar{u} is an isomorphism.

It is easily seen that, if $u \in \text{Mor}_{\mathcal{C}}(A, B)$ is arbitrary, then $\text{Ker}(u) \xrightarrow{i} A$ and $B \xrightarrow{j} \text{Coker}(u)$ are strict morphisms.

For the rest of our work we shall use the following convention: if \mathcal{C} is a semiabelian category, $A \in \text{Ob}(\mathcal{C})$ and (A', i') , (A'', i'') are subobjects of A (resp. (Q', p') , (Q'', p'') are quotients of A), the symbol $A' = A''$ (resp. $Q' = Q''$) will mean that A' and A'' (resp. Q' and Q'') are isomorphic.

Proposition 4.7. *Let \mathcal{C} be a semiabelian category and $A \in \text{Ob}(\mathcal{C})$. Then the following assertions hold:*

- (a) *if (A', i') is a strict subobject of A and $(\text{Coker}(i'), j')$ is the cokernel of i' , then $\text{Ker}(j') = A'$;*
- (b) *if (Q', p') is a strict quotient of A and $(\text{Ker}(p'), i')$ is the kernel of p' , then $\text{Coker}(i') = Q'$.*

Proof. (a): Let us consider the sequence

$$A' \xrightarrow{\ell} \text{Coim}(i') \xrightarrow{\overline{i'}} \text{Im}(i').$$

By Proposition 3.30(b), ℓ is an isomorphism; and, by hypothesis, $\overline{i'}$ is an isomorphism. Thus the morphism $\overline{i'}\ell$ is an isomorphism. Consequently,

$$A' = \text{Im}(i') = \text{Ker}(\text{Coker}(i')) = \text{Ker}(j').$$

(b): Follows from (a), by duality.

Definition 4.8. Let \mathcal{C} be a semiabelian category and $A \in \text{Ob}(\mathcal{C})$. For each strict subobject (A', i') of A , we shall write $\text{Coker}(i') = A/A'$.

The proof of the next result follows the lines of that of Theorem 2.13 of [12].

Proposition 4.9. *Let \mathcal{C} be a semiabelian category and $A \in \text{Ob}(\mathcal{C})$. Then any pair (A', i') , (A'', i'') of strict subobjects of A admits an infimum (denoted by $A' \cap A''$) in the preordered class of strict subobjects of A .*

Proof. Put $\text{Coker}(i') = (A/A', j')$ and $v = j'i'' : A'' \rightarrow A/A'$, and let $(\text{Ker}(v), i)$ be the kernel of v . Then $(\text{Ker}(v), i''i)$ is a strict subobject of A by Lemma 6 of [18], and $\text{Ker}(v) \leq A''$. We claim that $\text{Ker}(v) \leq A'$. In fact, since

$$0_{\text{Ker}(v)A/A'} = vi = (j'i'')i = j'(i''i)$$

and since $\text{Ker}(j') = A'$ by Proposition 4.7(a), there is a morphism $w : \text{Ker}(v) \rightarrow A'$ in \mathcal{C} such that the diagram

$$\begin{array}{ccc} \text{Ker}(v) & \xrightarrow{w} & A' \\ i \downarrow & & \downarrow i' \\ A'' & \xrightarrow{i''} & A \end{array}$$

is commutative by Proposition 3.5. Thus $\text{Ker}(v) \leq A'$.

Now, let (X, k) be a strict subobject of A such that $X \leq A'$ and $X \leq A''$. We claim that $X \leq \text{Ker}(v)$. Indeed, since $X \leq A'$, there is a morphism $\theta_1 : X \rightarrow A'$ in \mathcal{C} making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta_1} & A' \\ k \searrow & & \swarrow i' \\ & A & \end{array}$$

commutative. And, since $X \leq A''$, there is a morphism $\theta_2 : X \rightarrow A''$ in \mathcal{C} making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta_2} & A'' \\ k \searrow & & \swarrow i'' \\ & A & \end{array}$$

commutative. On the other hand, since

$$v\theta_2 = (j'i'')\theta_2 = j'(i''\theta_2) = j'k = j'(i'\theta_1) = (j'i')\theta_1 = 0_{A'A/A'}\theta_1 = 0_{XA/A'}$$

and since $(\text{Ker}(v), i)$ is the kernel of v , Proposition 3.5 guarantees the existence of morphism $t : X \rightarrow \text{Ker}(v)$ in \mathcal{C} making the diagram

$$\begin{array}{ccc} X & \xrightarrow{t} & \text{Ker}(v) \\ \theta_2 \searrow & & \swarrow i \\ & A'' & \end{array}$$

commutative. Consequently,

$$k = i''\theta_2 = i''(it) = (i''i)t,$$

proving that $X \leq \text{Ker}(v)$. This completes the proof.

Corollary 4.10. *Let \mathcal{C} be a semiabelian category and $A \in \text{Ob}(\mathcal{C})$. Then the following assertions hold:*

- (a) *every pair (Q', p') , (Q'', p'') of strict quotients of A admits an infimum (denoted by $Q' \cap Q''$) and a supremum (denoted by $Q' \cup Q''$) in the preordered class of strict quotients of A ;*
- (b) *every pair (A', i') , (A'', i'') of strict subobjects of A admits a supremum (denoted by $A' \cup A''$) in the preordered class of strict subobjects of A .*

Now we can state the following

Theorem 4.11. *Let \mathcal{C} be a semiabelian category and $A \in \text{Ob}(\mathcal{C})$, and let (A', i') , (A'', i'') be strict subobjects of A . Then there exists a bijective morphism*

$$A''/(A' \cap A'') \longrightarrow (A' \cup A'')/A'$$

in \mathcal{C} .

In order to prove Theorem 4.11 we shall need two auxiliary results.

Lemma 4.12. *Let \mathcal{C} be a semiabelian category and let $u \in \text{Mor}_{\mathcal{C}}(A, B)$ be such that $\text{Ker}(u) = A$. Then $u = 0_{AB}$.*

Proof. We have $\text{Ker}(u) \xrightarrow{i} A \xrightarrow{u} B$, with $ui = 0_{\text{Ker}(u)B}$, i being an isomorphism by hypothesis. Therefore

$$0_{AB} = (ui)i^{-1} = u(ii^{-1}) = u.$$

Lemma 4.13. *Let \mathcal{C} , A , (A', i') and (A'', i'') be as in Theorem 4.11, and consider the sequence*

$$A'' \xrightarrow{k} A' \cup A'' \xrightarrow{\ell} (A' \cup A'')/A'.$$

Then the morphism $u = \ell k$ is surjective.

Proof. Let $w: (A' \cup A'')/A' \rightarrow C$ be a morphism in \mathcal{C} such that $u_*(w) = wu = (w\ell)k = 0_{A''C}$. We have to show that $w = 0_{(A' \cup A'')/A' C}$ (Remark 3.3). But, since ℓ is surjective, it is enough to show that $w\ell = 0_{A' \cup A'' C}$. Indeed, by Proposition 4.5, $\text{Im}(k) \leq \text{Ker}(w\ell)$. On the other hand, by Proposition 4.7(a),

$$\text{Im}(k) = \text{Ker}(\text{Coker}(k)) = A'';$$

thus $A'' \leq \text{Ker}(w\ell)$. Moreover, by Propositions 4.7(a) and 3.32(a), $A' = \text{Ker}(\ell) \leq \text{Ker}(w\ell)$. Consequently, $A' \cup A'' \leq \text{Ker}(w\ell)$. But, since $\text{Ker}(w\ell) \leq A' \cup A''$, we get $\text{Ker}(w\ell) = A' \cup A''$. Therefore, by Lemma 4.12, $w\ell = 0_{A' \cup A'' C}$.

Now, let us turn to the

Proof of Theorem 4.11. We may assume that $A = A' \cup A''$. Let u be as in the proof of Lemma 4.13. As we have seen in the proof of Proposition 4.9, $\text{Ker}(u) = A' \cap A''$; thus

$$\text{Coim}(u) = \text{Coker}(\text{Ker}(u)) = A''/(A' \cap A'').$$

On the other hand, the morphism

$$A''/(A' \cap A'') \xrightarrow{\bar{u}} \text{Im}(u)$$

is bijective. Finally, by Lemma 4.13 and Proposition 3.30(d), $\text{Im}(u) = (A' \cup A'')/A'$, which concludes the proof.

Remark 4.14. The bijective morphism $A''/(A' \cap A'') \rightarrow (A' \cup A'')/A'$ considered in the proof of Theorem 4.11 is not necessarily an isomorphism; for instance, consider the semiabelian category of abelian topological groups and the example given in [4], p.27.

In order to prove a version of the first isomorphism theorem for semiabelian categories (see also [26]) we shall need the following

Proposition 4.15. *Let \mathcal{C} be a semiabelian category and $A \in \text{Ob}(\mathcal{C})$, and let (A', i') , (A'', i'') be strict subobjects of A such that $A' \leq A''$. Then there exists a unique strict surjective morphism $v: A/A' \rightarrow A/A''$ in \mathcal{C} making the diagram*

$$\begin{array}{ccc} & A & \\ j' \swarrow & & \searrow j'' \\ A/A' & \xrightarrow{v} & A/A'' \end{array}$$

commutative, where $j' = \text{Coker}(i')$ and $j'' = \text{Coker}(i'')$.

Proof. By hypothesis, there is a morphism $w: A' \rightarrow A''$ in \mathcal{C} such that $i' = i''w$ and, by Proposition 4.7(a), $\text{Ker}(j') = A'$ and $\text{Ker}(j'') = A''$. On the other hand, since

$$j''(i''w) = (j''i'')w = 0_{A''/A/A''} w = 0_{A'/A/A''},$$

Proposition 3.17 furnishes a morphism $v: A/A' \rightarrow A/A''$ in \mathcal{C} such that $j'' = vj'$. Since the surjectivity of v follows from assertion (b) after Definition 2.15 and the strictness of v follows from Lemma 4 of [18], the existence is proved.

To prove the uniqueness, let $v, v' \in \text{Mor}_{\mathcal{C}}(A/A', A/A'')$ be such that $j'' = vj'$ and $j'' = v'j'$. Then $vj' = v'j'$, and the surjectivity of j' gives $v = v'$.

Theorem 4.16. *Let \mathcal{C} be a semiabelian category and $A \in \text{Ob}(\mathcal{C})$. Let (A'', i'') be a strict subobject of A and (A', i') a strict subobject of A'' , and let $v: A/A' \rightarrow A/A''$ be as in Proposition 4.15. Then there exists a sequence*

$$A''/A' \longrightarrow \text{Ker}(v) \xrightarrow{\ell} A/A' \xrightarrow{v} A/A''$$

of morphisms in \mathcal{C} , the morphism on the left being bijective. Moreover, $(A/A')/\text{Ker}(v)$ and A/A'' are isomorphic.

Proof. Put $i' = i''i$. By Lemma 6 of [18], (A', i') is a strict subobject of A , and hence A/A' makes sense. Let α be the morphism $A'' \xrightarrow{i''} A \xrightarrow{j'} A/A'$ and consider the sequence

$$A' \xrightarrow{i} A'' \xrightarrow{\alpha} A/A' \xrightarrow{v} A/A''$$

of morphisms in \mathcal{C} . We claim that $\text{Im}(i) = \text{Ker}(\alpha)$ and $\text{Im}(\alpha) = \text{Ker}(v)$. Indeed, by Proposition 4.7(a), $\text{Im}(i) = \text{Ker}(\text{Coker}(i)) = A'$. And, since

$$\alpha i = (j'i'')i = j'(i''i) = j'i' = 0_{A'/A/A'},$$

Proposition 4.5 furnishes $\text{Im}(i) \leq \text{Ker}(\alpha)$. Let s be the injective morphism $\text{Ker}(\alpha) \rightarrow A'' \xrightarrow{i''} A$. Since the morphism

$$\text{Ker}(\alpha) \longrightarrow \underbrace{A'' \longrightarrow A \longrightarrow A/A'}_{\alpha}$$

is equal to $0_{\text{Ker}(\alpha)A/A'}$, that is, since the morphism

$$\text{Ker}(\alpha) \xrightarrow{s} A \xrightarrow{j'} A/A'$$

is equal to $0_{\text{Ker}(\alpha)A/A'}$, Proposition 3.5 implies the existence of a morphism $w: \text{Ker}(\alpha) \rightarrow A'$ in \mathcal{C} making the diagram

$$\begin{array}{ccc} \text{Ker}(\alpha) & \xrightarrow{w} & A' \\ s \searrow & & \swarrow i' \\ & A & \end{array}$$

commutative. Thus $\text{Ker}(\alpha) \leq A' = \text{Im}(i)$, and hence $\text{Im}(i) = \text{Ker}(\alpha)$. Therefore, by duality, $\text{Im}(\alpha) = \text{Ker}(v)$. Moreover, by hypothesis, the morphism $\bar{\alpha}: \text{Coim}(\alpha) \rightarrow \text{Im}(\alpha)$ is bijective, where

$$\text{Coim}(\alpha) = \text{Coker}(\text{Ker}(\alpha)) = A''/A'.$$

Finally, $\text{Coker}(\ell) = (A/A')/\text{Ker}(v)$ is isomorphic to A/A'' in view of Proposition 4.7(b).

This completes the proof.

Before proceeding we would like to mention that the proofs of Propositions 4.5 and 4.7 and of Theorem 4.16 are essentially contained in [2].

Remark 4.17. (a) Let $A \in \text{Ob}(\text{Mod}_R)$ and let $A' \subset A''$ be submodules of A (resp. $A \in \text{Ob}(\text{Topm}_R)$ and let $A' \subset A''$ be submodules of A endowed with the induced topology, $A \in \text{Ob}(\text{Ltm}_R)$ and let $A' \subset A''$ be submodules of A endowed with the induced topology, $A \in \text{Ob}(\text{Lcs})$ and let $A' \subset A''$ be subspaces of A endowed with the induced topology, $A \in \text{Ob}(\text{Lcs}_K)$ and let $A' \subset A''$ be subspaces of A endowed with the induced topology, $A \in \text{Ob}(\text{Borm}_R)$ and let $A' \subset A''$ be submodules of A endowed with the induced bornology, $A \in \text{Ob}(\text{Cbvs})$ and let $A' \subset A''$ be subspaces of A endowed with the induced bornology, $A \in \text{Ob}(\text{Cbvs}_K)$ and let $A' \subset A''$ be subspaces of A endowed with the induced bornology). If $v: A/A' \rightarrow A/A''$ is as in Theorem 4.16, it is easily seen that $\text{Ker}(v) = A''/A'$, and therefore $(A/A')/(A''/A')$ is isomorphic to A/A'' .

(b) Let $A \in \text{Ob}(\text{Ahtg})$ and let $A' \subset A''$ be closed subgroups of A endowed with the induced topology (resp. $A \in \text{Ob}(\text{Ban})$ and let $A' \subset A''$ be closed subspaces of A endowed with the induced norm). If $v: A/A' \rightarrow A/A''$ is as in Theorem 4.16, it is easily seen that $\text{Ker}(v) = A''/A'$, and therefore $(A/A')/(A''/A')$ is isomorphic to A/A'' .

We shall conclude our paper with a few comments on the notion of an abelian category [12], introduced by Buchsbaum [7] and Grothendieck [14].

Definition 4.18. A category \mathcal{C} is said to be *abelian* if it is preadditive, satisfies axioms (d) and (e) and the following axiom:

(f1) for all $A, B \in \text{Ob}(\mathcal{C})$ and for all $u \in \text{Mor}_{\mathcal{C}}(A, B)$, the morphism \bar{u} is an isomorphism.

Since every isomorphism is bijective, it follows that every abelian category is semiabelian.

Remark 4.19. In view of Propositions 3.30(b),(d) and 3.31 and the fact that every isomorphism is bijective, axiom (f1) is equivalent to the following axiom:

(f2) (a) for every morphism u in \mathcal{C} , the morphism \bar{u} is bijective; (b) every bijective morphism in \mathcal{C} is an isomorphism.

Let us mention a few examples of abelian categories.

Example 4.20. The category Mod_R is abelian.

Example 4.21. The category of finite abelian groups is abelian.

Example 4.22. The category of finite abelian p -groups [20] is abelian.

Example 4.23. The category of vector bundles [19] is abelian.

Example 4.24. The category of sheaves of abelian groups over a topological space [15] is abelian.

Now, let us see some examples of semiabelian categories which are not abelian.

Example 4.25. Let K be a discrete field. Then the semiabelian category of topological vector spaces over K is not abelian.

In fact, let A be a vector space over K , with $A \neq \{0\}$, and let τ_1 (resp. τ_2) be the discrete topology (resp. the chaotic topology) on A . Then $x \in (A, \tau_1) \mapsto x \in (A, \tau_2)$ is a bijective morphism in the category of topological vector spaces over K which is not an isomorphism. Thus, by Remark 4.19, this category is not abelian.

Example 4.26. Let K be a discrete field. Then the semiabelian category of linearly topologized spaces over K is not abelian.

In fact, it suffices to argue as in the preceding example, by observing that (A, τ_1) and (A, τ_2) are linearly topologized spaces over K .

Example 4.27. The semiabelian category \mathbf{Ahtg} is not abelian.

In fact, it suffices to recall Example 2.16 and Remark 4.19.

Example 4.28. The semiabelian category \mathbf{Ban} is not abelian.

In fact, consider the Banach spaces $A = (\ell^1, \|\cdot\|_1)$ and $B = (c_0, \|\cdot\|_0)$, and let $u: A \rightarrow B$ be the linear mapping given by $u(x) = x$ for $x \in A$. Since $\|u(x)\|_0 \leq \|x\|_1$ for all $x \in A$, $u \in \mathbf{Mor}_{\mathbf{Ban}}(A, B)$. Moreover, since A is dense in B , it follows that u is a bijective morphism in \mathbf{Ban} . Finally, u is not an isomorphism, and Remark 4.19 implies that \mathbf{Ban} is not abelian.

Example 4.29. The semiabelian category \mathbf{Lcs} is not abelian.

In fact, let A be an infinite-dimensional normed space endowed with the locally convex topology τ defined by its norm, and let $\sigma(A, A')$ be the weak topology on A . Then $x \in (A, \tau) \mapsto x \in (A, \sigma(A, A'))$ is a bijective morphism in \mathbf{Lcs} which is not an isomorphism. Thus, by Remark 4.19, \mathbf{Lcs} is not abelian.

Example 4.30. The semiabelian category \mathbf{Lcs}_K is not abelian.

In fact, consider the vector space $A = K^{(\mathbb{N})}$ over K endowed with the following locally K -convex topologies τ_1 and τ_2 : τ_1 is the direct sum topology [33, p. 268] (which is not metrizable by Theorem 3.13 of [33]) and τ_2 is the topology on A induced by the product topology on $K^{\mathbb{N}}$ (which is obviously metrizable). Then $x \in (A, \tau_1) \mapsto x \in (A, \tau_2)$ is a bijective morphism in \mathbf{Lcs}_K which is not an isomorphism. Therefore, by Remark 4.19, \mathbf{Lcs}_K is not abelian.

Example 4.31. Let K be a discrete field. Then the semiabelian category of bornological vector spaces over K is not abelian.

In fact, let A be an infinite-dimensional vector space over K , and let \mathcal{B}_1 (resp. \mathcal{B}_2) be the vector bornology having the finite-dimensional subspaces of A as a fundamental

system of bounded sets (resp. the vector bornology consisting of all subsets of A). Then $x \in (A, \mathcal{B}_1) \mapsto x \in (A, \mathcal{B}_2)$ is a bijective morphism in the category of bornological vector spaces over K which is not an isomorphism. Therefore, by Remark 4.19, this category is not abelian.

Example 4.32. The semiabelian category Cbvs_K is not abelian if K is spherically complete.

In fact, consider (A, τ_1) and (A, τ_2) as in Example 4.30 and let \mathcal{B}_1 (resp. \mathcal{B}_2) be the K -convex bornology consisting of all τ_1 -bounded (resp. τ_2 -bounded) subsets of A . Then $x \in (A, \mathcal{B}_1) \mapsto x \in (A, \mathcal{B}_2)$ is a bijective morphism in Cbvs_K which is not an isomorphism (for the boundedness of $x \in (A, \mathcal{B}_2) \mapsto x \in (A, \mathcal{B}_1)$ would imply the continuity of $x \in (A, \tau_2) \mapsto x \in (A, \tau_1)$ in view of the remark after Theorem 4.30 of [33]). Therefore, by Remark 4.19, Cbvs_K is not abelian.

In view of [9] we can argue as in Examples 4.30 and 4.32 to conclude:

Example 4.33. The semiabelian category Cbvs is not abelian.

We close our work by deriving the well-known isomorphism theorems for abelian categories [2,8,12].

Theorem 4.34. *Let \mathcal{C} be an abelian category and $A \in \text{Ob}(\mathcal{C})$, and let (A', i') , (A'', i'') be subobjects of A . Then*

$$A''/(A' \cap A'') \quad \text{and} \quad (A' \cup A'')/A'$$

are isomorphic.

Proof. Follows immediately from Theorem 4.11 and Remark 4.19.

Theorem 4.35. *Let \mathcal{C} be an abelian category and $A \in \text{Ob}(\mathcal{C})$. If (A'', i'') is a subobject of A and (A', i) is a subobject of A'' , then*

$$(A/A')/(A''/A') \quad \text{and} \quad A/A''$$

are isomorphic.

Proof. Follows immediately from Theorem 4.16 and Remark 4.19.

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